# Logic

## Propositional Logic

MAXIMUM SATISFIABILITY ◄ **LO1**

INSTANCE: Set $U$ of variables, collection $C$ of disjunctive clauses of literals, where a literal is a variable or a negated variable in $U$.

SOLUTION: A truth assignment for $U$.

MEASURE: Number of clauses satisfied by the truth assignment.

*Good News:* Approximable within 1.2987 [Asano, Hori, Ono, and Hirata, 1997].

*Bad News:* APX-complete [Papadimitriou and Yannakakis, 1991].

*Comment:* Variation in which each clause has a nonnegative weight and the objective is to maximize the total weight of the satisfied clauses is also approximable within 1.2987 [Asano, Hori, Ono, and Hirata, 1997]. Generalization in which each clause is a disjunction of conjunctions of literals and each conjunction consists of at most $k$ literals, where $k$ is a positive constant, is still APX-complete [Papadimitriou and Yannakakis, 1991]. Admits a PTAS for 'planar' instances [Khanna and Motwani, 1996]. The corresponding minimization problem MINIMUM SATISFIABILITY is approximable within 2 [Bertsimas, Teo, and Vohra, 1996] and its variation in which each clause has a nonnegative weight and the objective is to minimize the total weight of the satisfied clauses is as hard to approximate as the unweighted version [Crescenzi, Silvestri, and Trevisan, 1996] .

*Garey and Johnson:* LO1

MAXIMUM $k$-SATISFIABILITY ◄ **LO2**

INSTANCE: Set $U$ of variables, collection $C$ of disjunctive clauses of at most $k$ literals, where a literal is a variable or a negated variable in $U$. $k$ is a constant, $k \geq 2$.

SOLUTION: A truth assignment for $U$.

MEASURE: Number of clauses satisfied by the truth assignment.

*Good News:* Approximable within $1/(1-2^{-k})$ if every clause consists of exactly $k$ literals [Johnson, 1974a].

*Bad News:* APX-complete [Papadimitriou and Yannakakis, 1991].

*Comment:* If $k = 3$, the problem is approximable within 1.249 [Trevisan, Sorkin, Sudan, and Williamson, 1996] and is approximable within 8/7 for satisfiable instances [Karloff and Zwick, 1997]. MAXIMUM $k$-SATISFIABILITY is not approximable within $1/(1-2^{-k}) - \varepsilon$ for any $\varepsilon > 0$ and $k \geq 3$, even if every clause consists of exactly $k$ literals [Håstad, 1997].

MAXIMUM 2-SATISFIABILITY is approximable within 1.0741 [Feige and Goemans, 1995], and is not approximable within 1.0476 [Håstad, 1997]. The weighted version of this problem is as hard to approximate as the unweighted version [Crescenzi, Silvestri, and Trevisan, 1996].

If every clause consists of exactly $k$ literals, the weighted version of the problem is as hard to approximate as the unweighted version [Crescenzi, Silvestri, and Trevisan, 1996]. Variation in which the number of occurrences of any literal is bounded by a constant $B$ for $B \geq 3$ is still APX-complete even for $k = 2$ [Papadimitriou and Yannakakis, 1991] and [Chap.8 of this book]; for $B = 6$ it is not approximable within 1.0014 [Berman and Karpinski, 1998].

Admits a PTAS if $|C| = \Theta(|U|^k)$ [Arora, Karger, and Karpinski, 1995]. Variation in which each clause is a Horn clause, i.e., contains at most one nonnegated variable, is APX-complete, even for $k = 2$ [Kohli, Krishnamurti, and Mirchandani, 1994].

*Garey and Johnson:* LO2 and LO5

**LO3** ► MINIMUM $k$-SATISFIABILITY

INSTANCE: Set $U$ of variables, collection $C$ of disjunctive clauses of at most $k$ literals, where a literal is a variable or a negated variable in $U$. $k$ is a constant, $k \geq 2$.

SOLUTION: A truth assignment for $U$.

MEASURE: Number of clauses satisfied by the truth assignment.

*Good News:* Approximable within $2(1 - 1/2^k)$ [Bertsimas, Teo, and Vohra, 1996].

*Bad News:* APX-complete for every $k \geq 2$ [Kohli, Krishnamurti, and Mirchandani, 1994].

*Comment:* Transformation from MAXIMUM 2-SATISFIABILITY. Variation in which each clause is a Horn clause, i.e., contains at most one nonnegated variable, is APX-complete, even for $k = 2$ [Kohli, Krishnamurti, and Mirchandani, 1994].

*Garey and Johnson:* LO2

**LO4** ► MAXIMUM NOT-ALL-EQUAL 3-SATISFIABILITY

INSTANCE: Set $U$ of variables, collection $C$ of disjunctive clauses of at most 3 literals, where a literal is a variable or a negated variable in $U$.

SOLUTION: A truth assignment for $U$ and a subset $C' \subseteq C$ of the clauses such that each clause in $C'$ has at least one true literal and at least one false literal.

MEASURE: $|C'|$.

*Good News:* Approximable within 1.138 [Kann, Lagergren, and Panconesi, 1996].

*Bad News:* APX-complete [Papadimitriou and Yannakakis, 1991]. Not approximable within 1.090 [Zwick, 1998].

*Comment:* Transformation from MAXIMUM 2-SATISFIABILITY. Approximable within 1.096 for satisfiable instances [Zwick, 1998]. MAXIMUM NOT-ALL-EQUAL SATISFYABILITY, without restrictions on the number of literals in a clause, is approximable within 1.380 [Andersson and Engebretsen, 1998a].

*Garey and Johnson:* LO3

MINIMUM 3DNF SATISFIABILITY ◄ **LO5**

INSTANCE: Set $U$ of variables, collection $C$ of conjunctive clauses of at most three literals, where a literal is a variable or a negated variable in $U$.

SOLUTION: A truth assignment for $U$.

MEASURE: Number of clauses satisfied by the truth assignment.

*Bad News:* Not in APX [Kolaitis and Thakur, 1994].

*Garey and Johnson:* LO8

MAXIMUM DISTINGUISHED ONES ◄ **LO6**

INSTANCE: Disjoint sets $X,Z$ of variables, collection $C$ of disjunctive clauses of at most 3 literals, where a literal is a variable or a negated variable in $X \cup Z$.

SOLUTION: Truth assignment for $X$ and $Z$ that satisfies every clause in $C$.

MEASURE: The number of $Z$ variables that are set to true in the assignment.

*Bad News:* NPOPB-complete [Kann, 1992b].

*Comment:* Transformation from MAXIMUM NUMBER OF SATISFIABLE FORMULAS [Panconesi and Ranjan, 1993]. Not approximable within $(|X|+|Z|)^{1-\varepsilon}$ for any $\varepsilon > 0$ [Jonsson, 1997]. MAXIMUM ONES, the variation in which all variables are distinguished, i.e., $|X| = \emptyset$, is also NPOPB-complete [Kann, 1992b], and is not approximable within $|Z|^{1-\varepsilon}$ for any $\varepsilon > 0$ [Jonsson, 1997]. MAXIMUM WEIGHTED SATISFIABILITY, the weighted version, in which every variable is assigned a nonnegative weight, is NPO-complete [Ausiello, D'Atri, and Protasi, 1981].

MINIMUM DISTINGUISHED ONES ◄ **LO7**

INSTANCE: Disjoint sets $X,Z$ of variables, collection $C$ of disjunctive clauses of at most 3 literals, where a literal is a variable or a negated variable in $X \cup Z$.

SOLUTION: Truth assignment for $X$ and $Z$ that satisfies every clause in $C$.

MEASURE: The number of $Z$ variables that are set to true in the assignment.

*Bad News:* NPOPB-complete [Kann, 1994b].

*Comment:* Transformation from MINIMUM INDEPENDENT DOMINATING SET. Not approximable within $(|X|+|Z|)^{1-\varepsilon}$ for any $\varepsilon > 0$ [Jonsson, 1997]. MINIMUM ONES, the variation in which all variables are distinguished, i.e., $X = \emptyset$, is also NPOPB-complete [Kann, 1994b], and is not approximable within $|Z|^{1-\varepsilon}$ for any $\varepsilon > 0$ [Jonsson, 1997]. MINIMUM ONES for clauses of 2 literals is approximable within 2 [Gusfield and Pitt, 1992]. MINIMUM WEIGHTED SATISFIABILITY, the weighted version, in which every variable is assigned a nonnegative weight, is NPO-complete [Orponen and Mannila, 1987]. Variations corresponding to three- and four-valued logics have also been considered [Errico and Rosati, 1995].

**LO8** ► MAXIMUM WEIGHTED SATISFIABILITY WITH BOUND

INSTANCE: Set $U$ of variables, boolean expression $F$ over $U$, a nonnegative bound $B \in N$, for each variable $u \in U$ a weight $w(u) \in N$ such that $B \leq \sum_{u \in U} w(u) \leq 2B$.

SOLUTION: A truth assignment for $U$, i.e., a subset $U' \subseteq U$ such that the variables in $U'$ are set to true and the variables in $U - U'$ are set to false.

MEASURE: $\sum_{v \in U'} w(v)$ if the truth assignment satisfies the boolean expression $F$ and $B$ otherwise.

*Good News:* Approximable within 2 [Crescenzi and Panconesi, 1991].

*Bad News:* APX-complete [Crescenzi and Panconesi, 1991].

*Comment:* Variation with $\sum_{u \in U} w(u) \leq (1 + 1/(|U|-1))B$ is PTAS-complete [Crescenzi and Panconesi, 1991].

**LO9** ► MAXIMUM NUMBER OF SATISFIABLE FORMULAS

INSTANCE: Set $U$ of variables, collection $C$ of 3CNF formulas.

SOLUTION: A truth assignment for $U$.

MEASURE: Number of formulas satisfied by the truth assignment.

*Bad News:* NPOPB-complete [Kann, 1992b].

*Comment:* Transformation from LONGEST INDUCED PATH. Not approximable within $|C|$.

**LO10** ► MINIMUM NUMBER OF SATISFIABLE FORMULAS

INSTANCE: Set $U$ of variables, collection $C$ of 3CNF formulas.

SOLUTION: A truth assignment for $U$.

MEASURE: Number of formulas satisfied by the truth assignment.

*Bad News:* NPOPB-complete [Kann, 1994b].

*Comment:* Transformation from MINIMUM DISTINGUISHED ONES. Not approximable within $|C|^{1-\varepsilon}$ for any $\varepsilon > 0$ [Kann, 1994b].

MINIMUM EQUIVALENCE DELETION ◂ **LO11**

INSTANCE: Set $U$ of variables, collection $C$ of equivalences over $U$.

SOLUTION: A truth assignment for $U$.

MEASURE: Number of equivalences not satisfied by the truth assignment.

*Good News:* Approximable within $O(\log|U|)$ [Garg, Vazirani, and Yannakakis, 1996].

*Bad News:* APX-hard [Garg, Vazirani, and Yannakakis, 1996].

*Comment:* The dual problem is approximable within 1.138 [Kann, 1994b].

MAXIMUM $k$-CONSTRAINT SATISFACTION ◂ **LO12**

INSTANCE: Set $U$ of variables, collection $C$ of conjunctive clauses of at most $k$ literals, where a literal is a variable or a negated variable in $U$, and $k$ is a constant, $k \geq 2$.

SOLUTION: A truth assignment for $U$.

MEASURE: Number of clauses satisfied by the truth assignment.

*Good News:* Approximable within $2^{k-1}$ [Trevisan, 1996].

*Bad News:* APX-complete [Berman and Schnitger, 1992].

*Comment:* Transformation from MAXIMUM 2-SATISFIABILITY.

Approximable within 1.165 but not within 1.111 for $k = 2$, and approximable within 2 but not within $2-\varepsilon$ for $k = 3$. There are also results of specific variations of the problem for $k \leq 3$ [Zwick, 1998]. Not approximable within $2^{0.09k}$ for large enough $k$ [Trevisan, 1996]. Not in APX when $k = \log|C|$ [Verbitsky, 1995]. When there are exactly $k$ variables in each clause and the number of clauses is $\Theta(|U|^k)$ the problem admits a randomized PTAS [Andersson and Engebretsen, 1998b]. A complete classification of the approximability of optimization problems derived from Boolean constraint satisfaction is contained in [Khanna, Sudan, and Williamson, 1997] and in [Khanna, Sudan, and Trevisan, 1997].

MINIMUM LENGTH EQUIVALENT FREGE PROOF ◂ **LO13**

INSTANCE: A Frege proof $\pi$ of a tautology $\varphi$.

SOLUTION: A Frege proof $\pi'$ of $\varphi$ shorter than $\pi$, i.e., containing at most as many symbols as $\pi$.

MEASURE: Number of symbols in $\pi'$.

*Good News:* Approximable within $O(n)$.

*Bad News:* APX-hard [Alekhnovich, Buss, Moran, and Pitassi, 1998].

*Comment:* The result applies to all Frege systems, to all extended Frege systems, to resolution, to Horn clause resolution, to the sequent calculus, and to the cut-free sequent calculus. Not approximable within $2^{\log^{(1-\varepsilon)} n}$ for any $\varepsilon > 0$ unless NP $\subseteq$ QP [Alekhnovich, Buss, Moran, and Pitassi, 1998].

## Miscellaneous

**LO14** ► MAXIMUM HORN CORE

INSTANCE: Set $M$ of truth assignments on $n$ variables.

SOLUTION: A Horn core of $M$, i.e., a subset $M' \subseteq M$ such that $M'$ is equal to the set of truth assignments satisfying a Horn boolean formula.

MEASURE: The cardinality of the core, i.e., $|M'|$.

*Bad News:* Not in APX [Kavvadias, Papadimitriou, and Sideri, 1993].