DATA MINING 2

Naïve Bayes Classifiers

Riccardo Guidotti

a.a. 2021/2022

Slides edited from Tan, Steinbach, Kumar, Introduction to Data Mining

UNIVERSITÀ DI PISA

Bayes Classifier

- A probabilistic framework for solving classification problems.
- Let P be a probability function that assigns a number between 0 and 1 to events.
- $X = x$ an events is happening.
- $P(X = x)$ is the probability that events $X = x$.
- Joint Probability $P(X = x, Y = y)$
- Conditional Probability $P(Y = y \mid X = x)$
- Relationship: $P(X,Y) = P(Y|X)\ P(X) = P(X|Y)\ P(Y)$
- Bayes Theorem: $P(Y|X) = P(X|Y)P(Y) / P(X)$
- Another Useful Property: $P(X = x) = P(X=x, Y=0) + P(X=x, Y=1)$

Bayes Theorem

- Consider a football game. Team 0 wins 65% of the time, Team 1 the remaining 35%. Among the game won by Team 1, 75% of them are won playing at home. Among the games won by Team 0, 30% of them are won at Team 1's field.
- If Team 1 is hosting the next match, which team will most likely win?
- Team 0 wins: $P(Y = 0) = 0.65$
- Team 1 wins: $P(Y = 1) = 0.35$
- Team 1 hosted the match won by Team 1: $P(X = 1|Y = 1) = 0.75$
- Team 1 hosted the match won by Team 0: $P(X = 1|Y = 0) = 0.30$
- Objective $P(Y = 1|X = 1)$

Bayes Theorem

- $P(Y = 1|X = 1) = P(X = 1|Y = 1)P(Y = 1) / P(X = 1) =$
- $= 0.75 \times 0.35 / (P(X = 1, Y = 1) + P(X = 1, Y = 0))$
- $= 0.75 \times 0.35 / (P(X = 1|Y = 1)P(Y=1) + P(X = 1|Y = 0)P(Y=0))$
- $= 0.75 \times 0.35 / (0.75 \times 0.35 + 0.30 \times 0.65)$
- $= 0.5738$

- Therefore Team 1 has a better chance to win the match

Bayes Theorem for Classification

- X denotes the attribute sets, X = {X_1, X_2, … X_d}
- Y denotes the class variable
- We treat the relationship probabilistically using P(Y|X)

Likelihood

Prior Probability

- $\underbrace{P(Y|X)}_{\text{Posterior Probability}} = \dfrac{\overbrace{P(X|Y)}^{\text{Likelihood}}\overbrace{P(Y)}^{\text{Prior Probability}}}{\underbrace{P(X)}_{\text{Evidence}}}$

Posterior Probability

Evidence
(sum over alternative events)

Bayes Theorem for Classification

- Learn the posterior P(Y | X) for every combination of X and Y.
- By knowing these probabilities, a test record X’ can be classified by finding the class Y’ that maximizes the posterior probability P(Y’|X’).
- This is equivalent of choosing the value of Y’ that maximizes P(X’|Y’)P(Y’).
- How to estimate it?

Naïve Bayes Classifier

- It estimates the class-conditional probability by ***assuming that the attributes are conditionally independent*** given the class label y.
- The conditional independence is stated as:
- $P(X|Y = y) = \prod_{i=1}^{d} P(X_i|Y = y)$
- where each attribute set X = $\{X_1, X_2, \ldots X_d\}$

Conditional Independence

- Given three variables Y, X_1, X_2 we can say that Y is independent from X_1 given X_2 if the following condition holds:
- $P(Y \mid X_1, X_2) = P(Y|X_2)$

- With the conditional independence assumption, instead of computing the class-conditional probability for every combination of *X* we only have to estimate the conditional probability of each X_i given *Y*.
- Thus, to classify a record the naive Bayes classifier computes the posterior for each class *Y* and takes the maximum class as result
- $P(Y|X) = P(Y) \prod_{i=1}^{d} P(X_i|Y = y) \,/ P(X)$

How to estimate ?

How to Estimate Probability From Data

- Class $P(Y) = N_y / N$
- N_y number of records with outcome y
- N number of records
- Categorical attributes
- $P(X = x \mid Y = y) = N_{xy} / N_y$
- N_{xy} records with value x and outcome y

- P(Evade = Yes) = 3/10
- P(Marital Status = Single|Yes) = 2/3

Tid	Refund	Marital Status	Taxable Income	Evade
1	Yes	Single	125K	No
2	No	Married	100K	No
3	No	Single	70K	No
4	Yes	Married	120K	No
5	No	Divorced	95K	Yes
6	No	Married	60K	No
7	Yes	Divorced	220K	No
8	No	Single	85K	Yes
9	No	Married	75K	No
10	No	Single	90K	Yes

How to Estimate Probability From Data

Continuous attributes

- Discretize the range into bins
  - one ordinal attribute per bin
  - violates independence assumption
- Two-way split: $(X < v)$ or $(X > v)$
  - choose only one of the two splits as new attribute
- Probability density estimation:
  - Assume attribute follows a normal distribution
  - Use data to estimate parameters of distribution (e.g., mean and standard deviation)
  - Once probability distribution is known, can use it to estimate the conditional probability $P(X|y)$

How to Estimate Probability From Data

- Normal distribution
- $P(X_i = x_i \mid Y = y) = \frac{1}{\sqrt{2\pi}\sigma_{ij}} e^{-\frac{(x_i - \mu_{ij})^2}{2\sigma_{ij}^2}}$
- μ_{ij} can be estimated as the mean of X_i for the records that belongs to class y_j.
- Similarly, σ_{ij} as the standard deviation.
- P(Income = 120|No) = 0.0072
  - mean = 110
  - std dev = 54.54

Tid	Refund	Marital Status	Taxable Income	Evade
1	Yes	Single	125K	No
2	No	Married	100K	No
3	No	Single	70K	No
4	Yes	Married	120K	No
5	No	Divorced	95K	Yes
6	No	Married	60K	No
7	Yes	Divorced	220K	No
8	No	Single	85K	Yes
9	No	Married	75K	No
10	No	Single	90K	Yes

Example

Tid	Refund	Marital Status	Taxable Income	Evade
1	Yes	Single	125K	No
2	No	Married	100K	No
3	No	Single	70K	No
4	Yes	Married	120K	No
5	No	Divorced	95K	Yes
6	No	Married	60K	No
7	Yes	Divorced	220K	No
8	No	Single	85K	Yes
9	No	Married	75K	No
10	No	Single	90K	Yes

Given X = {Refund = No, Married, Income = 120k}

- P(Refund=Yes|No) = 3/7
- P(Refund=No|No) = 4/7
- P(Refund=Yes|Yes) = 0
- P(Refund=No|Yes) = 1
- P(Marital Status=Single|No) = 2/7
- P(Marital Status=Divorced|No)=1/7
- P(Marital Status=Married|No) = 4/7
- P(Marital Status=Single|Yes) = 2/3
- P(Marital Status=Divorced|Yes)=1/3
- P(Marital Status=Married|Yes) = 0/3

For taxable income:

- If class=No:
  - mean=110, variance=2975
- If class=Yes:
  - mean=90, variance=25

P(X|Class=No) = P(Refund=No|Class=No)
× P(Married| Class=No)
× P(Income=120K| Class=No)
= 4/7 × 4/7 × 0.0072
= 0.0024

P(X|Class=Yes) = P(Refund=No| Class=Yes)
× P(Married| Class=Yes)
× P(Income=120K| Class=Yes)
= 1 × 0 × 1.2 × 10-9
= 0

Since P(X|No)P(No) > P(X|Yes)P(Yes)

Therefore P(No|X) > P(Yes|X)
=> Class = No

M-estimate of Conditional Probability

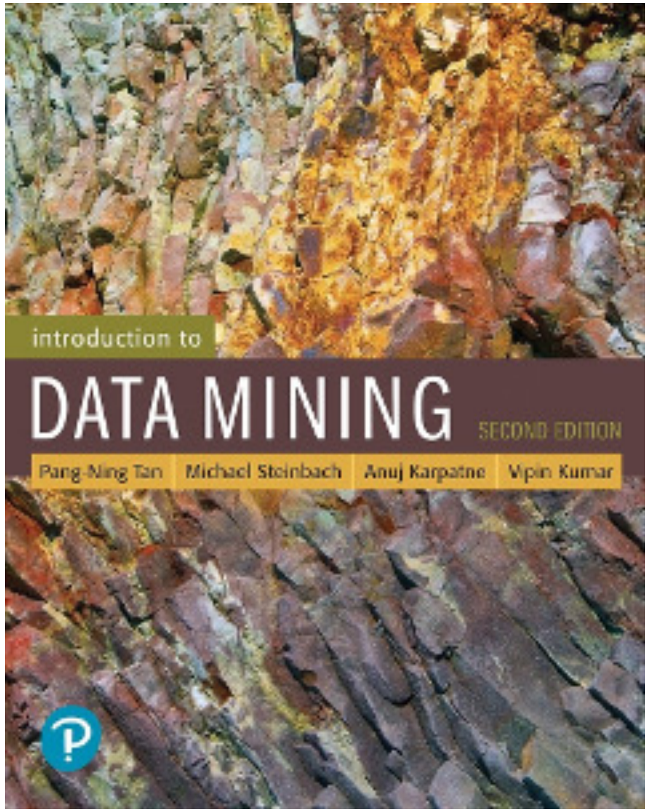
- If one of the conditional probability is zero, then the entire expression becomes zero.
- For example, given X = {Refund = Yes, Divorced, Income = 120k}, if P(Divorced|No) is zero instead of 1/7, then
  - P(X|No) = 3/7 x 0 x 0.00072 = 0
  - P(X|Yes) = 0 x 1/3 x 10^{-9} = 0
- M-estimate P(X|Y) = $\frac{N_{xy}+mp}{N_y+m}$ (if P(X|Y) = $\frac{N_{xy}+1}{N_y+|Y|}$ is Laplacian estimation)
- m is a parameter, p is a user-specified parameter (e.g. probability of observing x_i among records with class y_j.
- In the example with m = 3 and p = 1/m = 1/3 (i.e., Laplacian estimation) we have
- P(Married |Yes) = (0+3x1/3)/(3+3) = 1/6

Naïve Bayes Classifier

- Robust to isolated noise points
- Handle missing values by ignoring the instance during probability estimate calculations
- Robust to irrelevant attributes
- Independence assumption may not hold for some attributes
  - Use other techniques such as Bayesian Belief Networks (BBN, not treated in this course)

References

- Bayesian Classifiers. Chapter 5.3. Introduction to Data Mining.



Exercises - NBC

Play-tennis example. estimating $P(x_i|C)$

Outlook	Temperature	Humidity	Windy	Class
sunny	hot	high	false	N
sunny	hot	high	true	N
overcast	hot	high	false	P
rain	mild	high	false	P
rain	cool	normal	false	P
rain	cool	normal	true	N
overcast	cool	normal	true	P
sunny	mild	high	false	N
sunny	cool	normal	false	P
rain	mild	normal	false	P
sunny	mild	normal	true	P
overcast	mild	high	true	P
overcast	hot	normal	false	P
rain	mild	high	true	N

P(p) = 9/14
P(n) = 5/14

outlook	
P(sunny|p) =	P(sunny|n) =
P(overcast|p) =	P(overcast|n) =
P(rain|p) =	P(rain|n) =
temperature	
P(hot|p) =	P(hot|n) =
P(mild|p) =	P(mild|n) =
P(cool|p) =	P(cool|n) =
humidity	
P(high|p) =	P(high|n) =
P(normal|p) =	P(normal|n) =
windy	
P(true|p) =	P(true|n) =
P(false|p) =	P(false|n) =

Play-tennis example. estimating $P(x_i|C)$

Outlook	Temperature	Humidity	Windy	Class
sunny	hot	high	false	N
sunny	hot	high	true	N
overcast	hot	high	false	P
rain	mild	high	false	P
rain	cool	normal	false	P
rain	cool	normal	true	N
overcast	cool	normal	true	P
sunny	mild	high	false	N
sunny	cool	normal	false	P
rain	mild	normal	false	P
sunny	mild	normal	true	P
overcast	mild	high	true	P
overcast	hot	normal	false	P
rain	mild	high	true	N

$P(p) = 9/14$	
$P(n) = 5/14$	

outlook	
$P(sunny\|p) = 2/9$	$P(sunny\|n) = 3/5$
$P(overcast\|p) = 4/9$	$P(overcast\|n) = 0$
$P(rain\|p) = 3/9$	$P(rain\|n) = 2/5$
temperature	
$P(hot\|p) = 2/9$	$P(hot\|n) = 2/5$
$P(mild\|p) = 4/9$	$P(mild\|n) = 2/5$
$P(cool\|p) = 3/9$	$P(cool\|n) = 1/5$
humidity	
$P(high\|p) = 3/9$	$P(high\|n) = 4/5$
$P(normal\|p) = 6/9$	$P(normal\|n) = 1/5$
windy	
$P(true\|p) = 3/9$	$P(true\|n) = 3/5$
$P(false\|p) = 6/9$	$P(false\|n) = 2/5$

Play-tennis example. estimating $P(x_i|C)$

$P(p) = 9/14$	
$P(n) = 5/14$	

Outlook	Temeprature	Humidity	Windy	Class
rain	hot	high	false	?

outlook	
P(sunny|p) = 2/9	P(sunny|n) = 3/5
P(overcast|p) = 4/9	P(overcast|n) = 0
P(rain|p) = 3/9	P(rain|n) = 2/5
temperature	
P(hot|p) = 2/9	P(hot|n) = 2/5
P(mild|p) = 4/9	P(mild|n) = 2/5
P(cool|p) = 3/9	P(cool|n) = 1/5
humidity	
P(high|p) = 3/9	P(high|n) = 4/5
P(normal|p) = 6/9	P(normal|n) = 1/5
windy	
P(true|p) = 3/9	P(true|n) = 3/5
P(false|p) = 6/9	P(false|n) = 2/5

$P(X|p) \cdot P(p) =$

$P(X|n) \cdot P(n) =$

Play-tennis example. estimating $P(x_i|C)$

$P(p) = 9/14$	
$P(n) = 5/14$	

Outlook	Temeprature	Humidity	Windy	Class
rain	hot	high	false	N

outlook	
P(sunny|p) = 2/9	P(sunny|n) = 3/5
P(overcast|p) = 4/9	P(overcast|n) = 0
P(rain|p) = 3/9	P(rain|n) = 2/5
temperature	
P(hot|p) = 2/9	P(hot|n) = 2/5
P(mild|p) = 4/9	P(mild|n) = 2/5
P(cool|p) = 3/9	P(cool|n) = 1/5
humidity	
P(high|p) = 3/9	P(high|n) = 4/5
P(normal|p) = 6/9	P(normal|n) = 1/5
windy	
P(true|p) = 3/9	P(true|n) = 3/5
P(false|p) = 6/9	P(false|n) = 2/5

$P(X|p)\cdot P(p)$ = $P(rain|p)\cdot P(hot|p)\cdot P(high|p)\cdot P(false|p)\cdot P(p) = 3/9 \cdot 2/9 \cdot 3/9 \cdot 6/9 \cdot 9/14 = 0.010582$

$P(X|n)\cdot P(n)$ = $P(rain|n)\cdot P(hot|n)\cdot P(high|n)\cdot P(false|n)\cdot P(n) = 2/5 \cdot 2/5 \cdot 4/5 \cdot 2/5 \cdot 5/14 =$ 0.018286

Example of Naïve Bayes Classifier

Name	Give Birth	Can Fly	Live in Water	Have Legs	Class
human	yes	no	no	yes	mammals
python	no	no	no	no	non-mammals
salmon	no	no	yes	no	non-mammals
whale	yes	no	yes	no	mammals
frog	no	no	sometimes	yes	non-mammals
komodo	no	no	no	yes	non-mammals
bat	yes	yes	no	yes	mammals
pigeon	no	yes	no	yes	non-mammals
cat	yes	no	no	yes	mammals
leopard shark	yes	no	yes	no	non-mammals
turtle	no	no	sometimes	yes	non-mammals
penguin	no	no	sometimes	yes	non-mammals
porcupine	yes	no	no	yes	mammals
eel	no	no	yes	no	non-mammals
salamander	no	no	sometimes	yes	non-mammals
gila monster	no	no	no	yes	non-mammals
platypus	no	no	no	yes	mammals
owl	no	yes	no	yes	non-mammals
dolphin	yes	no	yes	no	mammals
eagle	no	yes	no	yes	non-mammals

A: attributes

M: mammals

N: non-mammals

$$P(A|M) = \frac{6}{7} \times \frac{6}{7} \times \frac{2}{7} \times \frac{2}{7} = 0.06$$

$$P(A|N) = \frac{1}{13} \times \frac{10}{13} \times \frac{3}{13} \times \frac{4}{13} = 0.0042$$

$$P(A|M)P(M) = 0.06 \times \frac{7}{20} = 0.021$$

$$P(A|N)P(N) = 0.004 \times \frac{13}{20} = 0.0027$$

Give Birth	Can Fly	Live in Water	Have Legs	Class
yes	no	yes	no	?

P(A|M)P(M) > P(A|N)P(N)

=> Mammals

a) Naive Bayes (**3 points**)

Given the training set below, build a Naive Bayes classification model (i.e. the corresponding table of probabilities) using (i) the normal formula and (ii) using Laplace formula. What are the main effects of Laplace on the models?

A	B	class
no	green	N
no	red	Y
yes	green	N
no	red	N
no	red	Y
no	green	Y
yes	green	N

Answer:

Normal

<table>
<tr><td></td><td>Y</td><td>N</td><td></td><td>Y</td><td>N</td></tr>
<tr><td></td><td>3</td><td>4</td><td></td><td>0.43</td><td>0.57</td></tr>
<tr><td></td><td>A | Y</td><td>A | N</td><td></td><td>A | Y</td><td>A | N</td></tr>
<tr><td>yes</td><td>0</td><td>2</td><td>yes</td><td>0.00</td><td>0.50</td></tr>
<tr><td>no</td><td>3</td><td>2</td><td>no</td><td>1.00</td><td>0.50</td></tr>
<tr><td></td><td>B | Y</td><td>B | N</td><td></td><td>B | Y</td><td>B | N</td></tr>
<tr><td>green</td><td>1</td><td>3</td><td>green</td><td>0.33</td><td>0.75</td></tr>
<tr><td>red</td><td>2</td><td>1</td><td>red</td><td>0.67</td><td>0.25</td></tr>
</table>

Laplace

<table>
<tr><td></td><td>Y</td><td>N</td><td></td><td>Y</td><td>N</td></tr>
<tr><td></td><td>3</td><td>4</td><td></td><td>0.43</td><td>0.57</td></tr>
<tr><td></td><td>A | Y</td><td>A | N</td><td></td><td>A | Y</td><td>A | N</td></tr>
<tr><td>yes</td><td>0</td><td>2</td><td>yes</td><td>0.20</td><td>0.50</td></tr>
<tr><td>no</td><td>3</td><td>2</td><td>no</td><td>0.80</td><td>0.50</td></tr>
<tr><td></td><td>B | Y</td><td>B | N</td><td></td><td>B | Y</td><td>B | N</td></tr>
<tr><td>green</td><td>1</td><td>3</td><td>green</td><td>0.40</td><td>0.67</td></tr>
<tr><td>red</td><td>2</td><td>1</td><td>red</td><td>0.60</td><td>0.33</td></tr>
</table>

a) Naive Bayes (**3 points**)

Given the training set on the left, build a Naive Bayes classification model and apply it to the test set on the right.

SCORE	FIRST-TRY	FACULTY	class
good	no	science	Y
medium	yes	science	N
bad	yes	science	N
bad	yes	humanities	Y
good	no	humanities	N
good	no	science	Y
medium	no	humanities	Y

SCORE	FIRST-TRY	FACULTY	class
bad	no	humanities	
good	yes	science	
medium	yes	humanities	